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$SO(4, 2)$-Covariant Formulation of the Bargmann–Wigner Equations

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Abstract

It is shown that the Bargmann–Wigner equations can be written in an $SO(4, 2)$-covariant form. As well as the Lorentz rotations, the $SO(4, 2)$ group contains a space-inversion and a time-reflection operator (which are different from the usual ones). It also contains the Foldy–Wouthuysen and Cini–Touschek transformations. The spin-s theory for the massive and massless cases, and also a set of Bargmann–Wigner equations corresponding to space-like four-momentum, are all given by the same $SO(4, 2)$-covariant equations, and their solutions can be obtained by transforming the solutions corresponding to the special "gauge" in which the four-momentum vanishes.

1. *Introduction*

The intimate connection between the group $SO(4, 1)$ and the structure of relativistic wave equations is well known from the work of Bhabha (1945). The wave functions of the equations proposed by Bhabha belong to irreducible representations of the "de Sitter group" $SO(4, 1)$, but the equations themselves are covariant only under the Lorentz subgroup. The de Sitter group plays only a formal role, facilitating the discussion of the algebra of the matrices that take the place of Dirac matrices in higher-spin theories, and aiding in the evaluation of their matrix elements. The de Sitter group plays a similar role in the work of de Vos and Hilgevoord (1967), in which a five-dimensional formalism was used to simplify the discussion of subsidiary conditions.

A different approach (Bakri, 1967, 1969, 1970; Bakri et al., 1970), in which the rest mass is regarded as a fifth component of momentum, leads to manifestly $SO(4, 1)$-covariant formulations of spin-s theories.

The wave functions of the Bargmann-Wigner equations (Bargmann and Wigner, 1946) belong to completely symmetric irreducible representations of the group $SO(4, 2)$, not just $SO(4, 1)$. The present work investigates the manifestly $SO(4, 2)$-covariant equations obtained by introducing an additional parameter so that the momentum vector acquires six components. The spin-

1/2 and spin-1 equations belonging to this scheme have been mentioned previously (Lord, 1972). We shall employ the algebraic methods of this work (referred to in the following as *I*), discussing the spin-1/2 and spin-1 theories in more detail and extending the discussion to spin-*s*.

The fact that the completely symmetric spinors of Bargmann-Wigner theory belong to irreducible representations of $SO(4, 2)$ has been used in the work of Salam et al. (1965). The group $SU(2, 2)$ of this work is the "universal covering group" or "spin-group" of $SO(4, 2)$. These authors obtained generalized Bargmann-Wigner equations in which the spin was not unique, by allowing irreducible representations other than the completely symmetric ones [cf, the spin-(1 + 0) of *I*]. However, as in the Bhabha equations, the equations themselves were covariant only under the Lorentz subgroup of the larger group used for classification of the wave functions.

2. *The Dirac Equation*

The 15 traceless base elements of the Dirac algebra are generators of a representation of the Lie algebra of $SO(4, 2)$ (ten Kate, 1968). The σ-algebra described in *I* (see also Lord, 1975) provides a method of formulating $SO(4, 2)$-covariant statements, analogous to the use of quaternion algebra in discussing $SO(3)$. The six matrices

$$\sigma^A = (\gamma^\mu, \gamma^5, 1) \qquad (A = 1, \ldots, 6) \tag{2.1}$$

satisfy

$$\sigma^{(A}\bar{\sigma}^{B)} = \bar{\sigma}^{(A}\sigma^{B)} = \eta^{AB} \tag{2.2}$$

where η^{AB} is the diagonal matrix $(111-1-11)$(used as a raising and lowering operator for sixfold indices), and

$$\bar{\sigma}^A = (-\gamma^\mu, -\gamma^5, 1) \tag{2.3}$$

The generators of the two spinor representations S and $\bar{S}$ of $SO(4, 2)$ are $\frac{1}{2}\sigma^{AB}$ and $\frac{1}{2}\bar{\sigma}^{AB}$, where

$$\sigma^{AB} = \sigma^{[A}\bar{\sigma}^{B]}, \qquad \bar{\sigma}^{AB} = \bar{\sigma}^{[A}\sigma^{B]} \tag{2.4}$$

The matrices S and $\bar{S}$ belong to $SU(2, 2)$. If the spinor ψ transforms to $S\psi$ under the six-dimentional "rotations," then the equation

$$(\bar{\sigma}^A p_A)\psi = 0 \tag{2.5}$$

is covariant. We wish to interpret p_μ as a four-momentum. Therefore, in order to preserve its reality under $SO(4, 2)$ transformations, we require the paramaters p_5 and p_6 to be real. In terms of the Dirac algebra, (2.5) is

$$(\gamma^\mu p_\mu + \gamma^5 p_5 + p_6)\psi = 0 \tag{2.6}$$

This implies a Klein-Gordon equation with $m^2 = p_6{}^2 - p_5{}^2$ [i.e. $(p^A p_A)\psi = 0$], so it is just as acceptable as the Dirac equation as a theory of massive spin-1/2 particles. The sign of $(p_6{}^2 - p_5{}^2)$ is not invariant under $SO(4, 2)$, so the theories of massive and massless spin-1/2, as well as a "tachyon" theory in which the four-momentum is spacelike, are contained in the single equation (2.5). In particular, we can obtain (2.5) in three special forms, connected by $SO(4, 2)$ rotations:

$$p_6 = \kappa,\ p_5 = 0:\ (\gamma^\mu p_\mu - \kappa)\psi = 0 \tag{2.7}$$

$$p_6 = 0,\ p_5 = \kappa:\ (\alpha^\mu p_\mu - i\kappa)\psi = 0 \tag{2.8}$$

$$p_6 = \pm p_5 = \kappa:\ [\gamma^\mu p_\mu + \kappa(1 \pm \gamma^5)]\,\psi = 0 \tag{2.9}$$

The first form is the conventional Dirac equation, which can be obtained from (2.6) by a rotation through angle $\tan^{-1}(p_5/p_4)$ in the (45)-plane. The second form corresponds to spacelike four-momentum – it is expressed in terms of a different representation of the Dirac matrices:

$$\alpha^\mu = -i\gamma^5\gamma^\mu = U\gamma^\mu U^{-1} \qquad (U = e^{-i\gamma^5/2}) \tag{2.10}$$

The third form corresponds to zero rest mass. It is identical to Tokuoka's neutrino equation (Tokuoka, 1967).

The generators σ^{AB} are explicitly

$$\begin{aligned} \sigma^{\mu\nu} &= -\tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \\ \sigma^{\mu 5} &= \gamma^5\gamma^\mu \\ \sigma^{\mu 6} &= \gamma^\mu \\ \sigma^{56} &= \gamma^5 \end{aligned} \tag{2.11}$$

The 16 Dirac bilinears consist of a scalar $\tilde{\psi}\psi$ and a skew tensor $\tilde{\psi}\sigma^{AB}\psi$ of $SO(4, 2)$ (the skew tensor belonging of course to the adjoint representation), where

$$\tilde{\psi} = i\bar{\psi}\gamma^5 = i\psi^\dagger\beta\gamma^5 \tag{2.12}$$

It is rather surprising that the pseudoscalar is the invariant for $SO(4, 2)$, rather than the scalar $\bar{\psi}\psi$. This is, however, an essential feature of the formalism. Some clarification is required here, since the relation between the notation of *I* and that of the present work is somewhat confusing. In *I*, the representation of the Dirac algebra that we employed was the α^μ of (2.10). The "Dirac equation" of that paper was actually the tachyon equation (2.8), but since the factor i was inadvertently omitted, this fact was overlooked. The quantity $\tilde{\psi}$ is actually the adjoint of ψ associated with the Dirac matrices α^μ, and the matrix β and charge conjugation matrix C of *I* were also those appropriate to the Dirac matrices α^μ. The relation between the notation of the present paper and that of *I* is explicitly given by (2.10), together with $\alpha^5 = \gamma^5$, $\beta_I = i\gamma^5\beta$, $\bar{\psi}_I = \tilde{\psi}$, $C_I = i\gamma_5 C$.

Multiplying (2.5) by C_I gives it the form

$$p^A\sigma_A{}^{\alpha\beta}\psi_\beta = 0 \tag{2.13}$$

where $\sigma_A{}^{\alpha\beta}$ is skew-symmetric in its spinor indices.

3. *Some Explicit SU(2, 2) Matrices*

A parity transformation can be regarded as an inversion in (1236)-space, and time-reversal as an inversion in the (45)-plane. The corresponding spinor matrices are

$$P = e^{\pi\sigma^{12}/2}\, e^{\pi\sigma^{36}/2} = (-\gamma^1\gamma^2)\gamma^3 = i\gamma^5\beta \tag{3.1}$$

and

$$T = e^{\pi\sigma^{45}/2} = \gamma^5\beta \tag{3.2}$$

They are both unitary. These parity and time-reversal operations are different from the usual ones. In particular, T operates linearly on the wave function, not "antilinearly," and under both P and T, the Dirac bilinear $\bar{\psi}\psi$ changes sign, while $\bar{\psi}\gamma^5\psi$ remains invariant, the opposite of the usual behavior. Moreover, if we take the spinor equation in the form (2.7), the "parity" operation involves a change on the sign of the mass, $m \to -m$. The physical significance of these operations P and T is at present obscure.

A pure Lorentz transformation is a hyperbolic rotation in the (p4)-plane (where p denotes the axis along the three-momentum $\mathbf{p}$). The transformation to the rest frame is the one that takes $p = |\mathbf{p}|$ to zero. However, the group $SO(4, 2)$ contains another transformation that will take $\mathbf{p}$ to zero, namely, a rotation through angle

$$\theta_1 = \tan^{-1}(p/p_6) \tag{3.3}$$

in the (p6)-plane. The spinor matrix of this transformation is

$$\exp\left(\tfrac{1}{2}\theta_1 p_i \sigma^{i5}/p\right) = \exp\left(\tfrac{1}{2}\theta_1 \hat{\mathbf{p}}\cdot\boldsymbol{\Upsilon}\right) \tag{3.4}$$

If the equation is initially in the form of the Dirac equation (2.7), this is just the Foldy-Wouthuysen transformation (Foldy and Wouthuysen, 1950):

$$\frac{m + p_4 + \boldsymbol{\Upsilon}\cdot\mathbf{p}}{[2p_4(p_4 + m)]^{1/2}} \tag{3.5}$$

Alternatively, a rotation through angle

$$\theta_2 = \tan^{-1}(p_6/p) = \pi/2 - \theta_1 \tag{3.6}$$

in the (p6)-plane will take p_6 to zero. With the equation initially in the form (2.7), we obtain the Cini-Touschek transformation (Cini and Touschek, 1958).

The most simple special form for (2.5) that can be obtained by an $SO(4, 2)$ rotation, corresponds to p_A = (000011). The equation is then simply

$$\psi = \gamma^5\psi \tag{3.7}$$

In the Dirac representation, the solution is

$$\psi = \begin{pmatrix} \chi \\ \chi \end{pmatrix} \tag{3.8}$$

where χ is an arbitrary two-component spinor. We obtain the positive-energy solutions of the Dirac equation by a rotation of 90° in the (45)-plane $[p_A \to (000101), \psi \to \exp(-\frac{1}{4}\pi\gamma^5\gamma^4)\psi \sim \binom{\chi}{0}]$, followed by an inverse Foldy-Wouthuysen transformation. The negative-energy solutions are obtained by a rotation through 90° in the opposite direction $[p_A \to (000\text{-}101), \psi \to \exp(\frac{1}{4}\pi\gamma^5\gamma^4)\psi \sim \binom{0}{\chi}]$ followed by the inverse Foldy-Wouthuysen transformation. Solutions of the Tokuoka equation (2.9) are obtained from (3.7) by combining a rotation in the (p6)-plane with a rotation through the *same angle* in the (45)-plane.

4. *Spin*-1

The generalization of (2.13) to spin-1 is

$$p^A \sigma_A{}^{\alpha\beta} B_{\beta\gamma} = 0 \tag{4.1}$$

where $B_{\beta\gamma}$ is symmetric in its spinor indices (therefore ten components). The number of equations contained in (4.1) is 15 (*not* 16, because the equation obtained by contraction on $\alpha\gamma$ is an identity arising from the skew-symmetry of $\sigma_A{}^{\alpha\beta}$). The usual Bargmann-Wigner equations for spin-1 are, of course, (4.1) with $p_5 = 0, p_6 = \kappa$.

The equations (4.1) can be written in various alternative forms. The 10 x 10 matrices G^{AB} that generate the representation B of $SO(4, 2)$ are of course given by

$$(G_{AB}B)_{\alpha\beta} = \sigma_{AB(\alpha}^{\gamma} B_{\beta)\gamma} \tag{4.2}$$

Therefore, if we multiply (4.1) by $\sigma_{B\delta\alpha}$ and symmetrize on $\delta\gamma$ we get (on account of $\sigma_B \bar{\sigma}_A = \sigma_{AB} + \eta_{AB}$)

$$(G_{BA}p^A + p_B)B = 0 \tag{4.3}$$

The equations (4.3) are actually *equivalent* to (4.1): Multiply (4.3) by σ^B. We obtain

$$0 = p^A \sigma^{B\alpha\beta} \sigma_{BA}^{\delta}{}_{(\beta} B_{\gamma)\delta} + p^B \sigma_B^{\alpha\beta} B_{\beta\gamma} \tag{4.4}$$

On account of the identities

$$\sigma_{BA\beta}{}^{\delta} = \sigma_{B\beta\rho} \sigma_A^{\rho\delta} - \eta_{AB} \delta_\beta^\delta$$

and

$$\sigma^{B\alpha\beta} \sigma_{B\gamma\delta} = 4\delta_{\delta\gamma}^{[\alpha\beta]}$$

equation (4.4) is

$$p^A \sigma_A^{\delta[\alpha} \delta^{\beta]}{}_{(\beta} B_{\gamma)\delta} = 0$$

When the symmetrization and skew-symmetrization brackets are expanded, this is seen to be equation (4.1). The equations (4.3) are, therefore, completely equivalent to the generalized Bargmann-Wigner equations (4.1).

As shown in *I*, the ten-dimensional representation B of $SO(4, 2)$ is a skew-symmetric anti-self-dual tensor of rank 3, In this formulation, the components of B are written with a skew-symmetric triplet of sixfold indices

$$B_{ABC} \tag{4.5}$$

and we have

$$B_{ABC} = -(i/6)\epsilon_{ABCDEF}B^{DEF} \tag{4.6}$$

The Bargmann–Wigner equations (4.1) are then

$$p_A B^{ABC} = 0 \tag{4.7}$$

Because of the anti-self-dual property (4.6), equations (4.7) can be written in the alternative (but equivalent) form

$$p_{[A}B_{BCD]} = 0 \tag{4.8}$$

The equations (4.7) can be written in a five-dimensional notation by singling out one of the momentum components. *Without loss of generality,* we single out p_6, and let the indices $a, b, c, \ldots$ run from 1 to 5. Define

$$B_{ab} = B_{ab6} \tag{4.9}$$

The wave function is now simply a rank-2 skew-symmetric tensor for the "little group" $SO(3,2)$ (if we had singled out p_5, the "little group" would have been $SO(4, 1)$, and if we had singled out $p_5 + p_6$, it would have been a group isomorphic to the Poincaré group). The equations (4.6) and (4.7) are now

$$B_{abc} = -\frac{i}{2}\epsilon_{abcde}B^{de}, \qquad B_{ab} = \frac{i}{6}\epsilon_{abcde}B^{cde} \tag{4.10}$$

$$p_6B^{ab} + p_cB^{cab} = 0, \qquad p_aB^{ab} = 0 \tag{4.11}$$

We shall refer to the equation (4.7) for particular values of B and C as "the (BC)-equation." Then the first set of equations (4.11) consists of the equations (ab) and the second set consists of the equations ($b6$). Clearly, *if p_6 is not zero, the five equations ($b6$) are a consequence of the ten equations (ab).* This removes the redundancy from the Bargmann–Wigner equations (4.1). The ten equations (ab) are explicitly

$$PB = 0 \tag{4.12}$$

where B is the ten-component vector $(B^{23}, B^{31}, B^{12}, B^{14}, B^{24}, B^{34}, B^{15}, B^{25}, B^{35}, B^{45}) = (B^{236}, B^{316}, B^{126}, B^{146}, B^{246}, B^{346}, B^{156}, B^{256}, B^{356}, B^{456}) = i(B_{145}, B_{245}, B_{345}, B_{235}, B_{315}, B_{125}, B_{325}, B_{324}, B_{134}, B_{214}, B_{321})$ and P is the matrix

$$\left|\begin{array}{cccccccccc}
p_6 & \cdot & \cdot & ip_5 & \cdot & \cdot & -ip_4 & \cdot & \cdot & -ip_1 \\
\cdot & p_6 & \cdot & \cdot & ip_5 & \cdot & \cdot & -ip_4 & \cdot & -ip_2 \\
\cdot & \cdot & p_6 & \cdot & \cdot & ip_5 & \cdot & \cdot & -ip_4 & -ip_3 \\
-ip_5 & \cdot & \cdot & p_6 & \cdot & \cdot & \cdot & -ip_3 & ip_2 & \cdot \\
\cdot & -ip_5 & \cdot & \cdot & p_6 & \cdot & ip_3 & \cdot & -ip_1 & \cdot \\
\cdot & \cdot & -ip_5 & \cdot & \cdot & p_6 & -ip_2 & ip_1 & \cdot & \cdot \\
ip_4 & \cdot & \cdot & \cdot & ip_3 & -ip_2 & p_6 & \cdot & \cdot & \cdot \\
\cdot & ip_4 & \cdot & -ip_3 & \cdot & ip_1 & \cdot & p_6 & \cdot & \cdot \\
\cdot & \cdot & ip_4 & ip_2 & -ip_1 & \cdot & \cdot & \cdot & p_6 & \cdot \\
-ip_1 & -ip_2 & -ip_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p_6
\end{array}\right| \tag{4.13}$$

Returning now to the formulation (4.3), define

so that

$$\left.\begin{aligned} \beta_a &= G_{a6} \\ [\beta_a, \beta_b] &= G_{ab} \end{aligned}\right\} \tag{4.14}$$

The matrices β^a are then the ten-dimensional representation of the Kemmer algebra K_5 (Lord, 1973),

$$\beta^a\beta^b\beta^c + \beta^c\beta^b\beta^a = \eta^{ab}\beta^c + \eta^{cb}\beta^a \tag{4.15}$$

The sixth component of the set of equations (4.3) is then

$$(-\beta^a p_a + p_6)B = 0 \tag{4.16}$$

which is clearly the same as equation (4.12). [The coefficient matrices of the p_a in (4.13) are the *five* Kemmer matrices.] Hence *one* of the six components of the set of equations (4.3) implies the remaining five. Note that the choice of p_6 as the component singled out for special attention has no particular significance; in particular, the indices 123456 on B_{ABC} and p_A can be *permuted* so that any of the six components of p_A can appear on the diagonal in (4.13). Singling out p_4, the relevant component of (4.3) is $p_4 = G_{A4}p^A$; setting $p_5 = 0$, $p_6 = m$ we obtain the Hamiltonian form of the Kemmer equation,

$$p_4 = [\beta^4, \boldsymbol{\beta}] \cdot \mathbf{p} + \beta^4 m$$

The derivation of the Klein-Gordon equation ($p_A p^A B = 0$) from (4.3) is immediate. In particular, the equations (4.16) with $p_5 = \pm p_6$ describe zero rest-mass spin-1 (i.e., they are equivalent to the Maxwell equations). A similar formulation of Maxwell's equations, obtained from the Kemmer equation by replacing the mass by a singular operator, was given by Harish-Chandra (1947).

The actual Harish-Chandra equations can be obtained from (4.12) by setting $p_5 = 0$, rewriting in terms of the ten-component wave function $(B^{\mu\nu}, B^{\mu 5}/p_6)$ instead of $(B^{\mu\nu}, B^{\mu 5})$, and then taking the limit as p_6 goes to zero. Such a procedure, however, is contrary to the spirit of the $SO(4, 2)$ formulation. Our singular operator is $1 \pm \beta_5$; Harish-Chandra's is $\beta_5{}^2$.

The action of the Foldy-Wouthuysen and Cini-Touschek transformations on the ten-component wave function are seen immediately, from the previous discussion, to be given by the matrices

$$\exp(\tfrac{1}{2}\theta\, \hat{\mathbf{p}} \cdot \boldsymbol{\beta}) = 1 + (\hat{\mathbf{p}} \cdot \boldsymbol{\beta}) \sin \tfrac{1}{2}\theta + (\hat{\mathbf{p}} \cdot \boldsymbol{\beta})^2 (1 - \cos \tfrac{1}{2}\theta) \tag{4.17}$$

with the appropriate value of θ.

5. *Spin-$\frac{3}{2}$*

The generalized Bargmann-Wigner equations for spin-$\frac{3}{2}$ are

$$p^A \sigma_A^{\alpha\beta} B_{\beta\gamma\delta} = 0 \tag{5.1}$$

where $B_{\beta\gamma\delta}$ is a completely symmetric spinor. The first index pair can be converted to an anti-self-dual triplet of sixfold indices, as in Section 4, so we have a spin-tensor $B_{ABC\delta}$ satisfying

$$p_A B^{ABC} = 0 \tag{5.2}$$

$$(\bar{\sigma}^A p_A) B^{CDE} = 0 \tag{5.3}$$

These equations must be supplemented by a subsidiary condition corresponding to symmetry of

$$B_{\beta\gamma\delta} = \tfrac{1}{12} \sigma_{\beta\gamma}^{ABC} B_{ABC\delta} \tag{5.4}$$

on $\gamma\delta$. Since $\sigma_D^{\gamma\delta}$ are a complete set of skew-symmetric matrices, this symmetry property is ensured by requiring the vanishing of (5.4) when multiplied by $\sigma_D^{\gamma\delta}$. The required supplementary condition is therefore

$$\sigma^{ABC} \bar{\sigma}^D B_{ABC} = 0 \tag{5.5}$$

This is readily simplified by using the rules given in *I* for evaluating products such as $\sigma^{ABC}\bar{\sigma}^D$. We find that (5.5) is simply

$$\sigma_{AB} B^{ABC} = 0 \tag{5.6}$$

An alternative form for the subsidiary condition is

$$\bar{\sigma}_A B^{ABC} = 0 \tag{5.7}$$

The equivalence of (5.6) and (5.7) is rather surprising. We show directly that (5.7) is equivalent to the symmetry of $B_{\beta\gamma\delta}$ on $\beta\delta$. Using the methods of *I*,

$$\sigma_A^{\alpha\beta} B_\beta^{ABC} = -\tfrac{1}{4} \sigma_A^{\alpha\beta} \sigma^{A\delta\rho} \sigma^{BC\delta}{}_\rho B_{\gamma\delta\beta} = -\tfrac{1}{2} \epsilon^{\alpha\beta\gamma\rho} \sigma^{BC\delta}{}_\rho B_{\gamma\delta\beta}$$

Since σ^{BC} is a complete set of traceless 4 x 4 matrices, the vanishing of this quantity is equivalent to the vanishing of $\epsilon^{\alpha\beta\gamma\rho}B_{\gamma\delta\rho}$, which is equivalent to $B_{\gamma\delta\rho} = B_{\rho\delta\gamma}$.

Obviously, when $B_{\alpha\beta\gamma}$ is completely symmetric, the two equations (5.2) and (5.3) are equivalent to each other. The spin-$\frac{3}{2}$ theory is therefore completely specified by the set of equations

$$p_A B^{ABC} = 0, \qquad \bar{\sigma}_A B^{ABC} = 0 \tag{5.8}$$

6. *Spin*-2

For spin-2, we have a completely symmetric rank-4 spinor. The two pairs of spinor indices can be converted to anti-self-dual triplets of tensor indices, so we have a wave function

$$B_{ABC,DEF} \tag{6.1}$$

symmetric under interchange of the triplets. Alternatively we could leave one of the spinor index pairs unconverted; in terms of the resulting mixed quantity, the remaining symmetries of the rank-4 spinor can be seen to be equivalent to

$$\bar{\sigma}_A^{\alpha\beta} B_{\beta\gamma}^{ABC} = \tfrac{1}{12}\bar{\sigma}_A^{\alpha\beta}\sigma_{\beta\gamma}^{DEF} B_{DEF}^{ABC} = 0$$

From the expression given in *I* for the product $\bar{\sigma}_A \sigma^{DEF}$, this is easily seen to be

$$B_{AEF}^{ABC} = 0 \tag{6.2}$$

Hence the Bargmann–Wigner spin-2 equations are equivalent to

$$p_A B_{DEF}^{ABC} = 0 \tag{6.3}$$

together with the supplementary condition (6.2).

7. *Higher Spin*

For integral spin s, the wave function has s anti-self-dual triplets. It is symmetric under interchange of triplets, and the contraction (6.2) on any pair of triplets vanishes. It satisfies equation (6.3). For half-integral spin we have exactly the same properties on $(s - \frac{1}{2})$ triplets, but there is an additional spinor index, and an additional subsidiary condition (5.7) associated with it. However, note that for half-integral spin greater than $\frac{3}{2}$ condition (6.2) is a consequence of the condition (5.7); for example, for spin-5/2 we have, from (5.7),

$$\sigma^A(\bar{\sigma}^D B_{ABC,DEF}) + \sigma^D(\bar{\sigma}^A B_{ABC,DEF}) = 2\eta^{AD} B_{ABC,DEF} = 0$$

which is just (6.2). Hence the *complete* sets of Bargmann–Wigner equations are

$$\begin{aligned} &p_A B^{ABC}_{\cdots} = 0, \qquad B^{ABC}_{AEF\cdots} = 0 \qquad \text{for integral spin} \\ &(\bar{\sigma}^A p_A) B^{BCD}_{\cdots} = 0, \qquad \bar{\sigma}_A B^{ABC}_{\cdots} = 0 \qquad \text{for half-integral spin} \end{aligned} \tag{7.1}$$

As for spin-1, the generators of $SO(4, 2)$ for a completely symmetric spinor are given by

$$(G_{AB}B)_{\alpha\beta\gamma\cdots} = \tfrac{1}{2}s\sigma^{\delta}_{AB(\alpha}B_{\beta\gamma\cdots)\delta}$$

so that, by multiplying

$$p^A \sigma_A^{\alpha\beta} B_{\beta\gamma\cdots} \tag{7.2}$$

by $\sigma_{B\delta\alpha}$ and symmetrizing on the remaining indices we get

$$(G_{BA}p^A + sp_B)B = 0 \tag{7.3}$$

[However, it is not in general true that (7.3) implies the full set (7.2); that is a peculiarity of spin-$\frac{1}{2}$ and spin-1.] Equation (4.14) now serves to define the *Bhabha* matrices, and the Foldy–Wouthuysen and Cini–Touschek transformations have the form

$$\exp\left(\tfrac{1}{2}\theta\, \hat{\mathbf{p}} \cdot \boldsymbol{\beta}\right) \tag{7.4}$$

The solutions for massive or massless spin-s can be obtained by $SO(4, 2)$ rotations from the solution in the special gauge with $p_A = (000011)$, as for spin-$\frac{1}{2}$. In this special gauge we have $B_{5BC} = B_{6BC}$, so $B_{\mu 56} = 0$ and therefore $B_{\mu\nu\rho} = 0$. Each triplet is therefore a self-dual skew-symmetric pair of *fourfold* indices: $B_{\mu\nu} = B_{\mu\nu 6}$ satisfying

$$B^{\mu\nu} = \tfrac{1}{2}i\epsilon^{\mu\nu\rho\sigma}B_{\rho\sigma} \tag{7.5}$$

The subsidiary conditions for integral spin are

$$B^{\mu\nu}_{\mu\rho,\ldots} = 0 \tag{7.6}$$

and those for half-integral spin are

$$\gamma^{\mu}B_{\mu\nu,\ldots} = 0, \qquad B_{\mu\nu,\ldots} = \gamma^5 B_{\mu\nu,\ldots} \tag{7.7}$$

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References

Bakri, M. M. (1967). *Nuovo Cimento,* **51**, 864.
Bakri, M. M. (1969). *Journal of Mathematical Physics,* **10**, 298.
Bakri, M. M. (1970). *Journal of Mathematical Physics,* **11**, 2027.
Bakri, M. M., Ghaleb, A. F. and Hessein, M. I. (1970). *Journal of Mathematical Physics,* **11**, 2015.
Bargmann, V. and Wigner, E. P. (1946). *Proceedings of the National Academy of Sciences,* **34**, 211.
Bhabha, H. J. (1945). *Reviews of Modern Physics,* **17**, 200.
Cini, M. and Touschek, B. (1958). *Nuovo Cimento,* **7**, 422.
Foldy, L. L. and Wouthuysen, S. A. (1950). *Physical Review,* **78**, 29.
Harish-Chandra (1947). *Proceedings of the Royal Society,* **A189**, 372.

ten Kate, A. (1968). *Nuclear Physics*, **B8**, 73.
Lord, E. A. (1972). *International Journal of Theoretical Physics*, **5**, 339.
Lord, E. A. (1973). *International Journal of Theoretical Physics*, **7**, 153.
Lord, E. A. (1975). *International Journal of Theoretical Physics*, **13**, 89.
Salam, A., Delbourgo, R. and Strathdee, J. (1965). *Proceedings of the Royal Society*, **A384**, 146.
Tokuoka, Z. (1967). *Progress of Theoretical Physics*, **37**, 603.
de Vos, J. A. and Hilgevoord, J. (1967). *Nuclear Physics*, **B1**, 494.